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THE ROLE OF DIMENSIONALITY IN THE STABILITY OF A CONFINED CONDENSED BOSE GAS

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Abstract

We study analytically the ground–state stability of a Bose–Einstein condensate (BEC) confined in an harmonic trap with repulsive or attractive zero–range interaction by minimizing the energy functional of the system. In the case of repulsive interaction the BEC mean radius grows by increasing the number of bosons, instead in the case of attractive interaction the BEC mean radius decreases by increasing the number of bosons: to zero if the system is one–dimensional and to a minimum radius, with a maximum number of bosons, if the system is three–dimensional.

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In the last three years there has been a renewed interest in the Bose-Einstein condensation due to the spectacular experiments with alkali vapors ^{87}Rb, ^{23}Na and ^{7}Li confined in magnetic traps and cooled down to a temperature of the order of 100 nK[1),2),3)]. Numerical studies of the Bose–Einstein condensate (BEC) have been performed for the ground state[4),5),6)] and the collective excitations[7),8),9)].

In the present paper we analyze analytically the ground–state stability of the BEC by minimizing the energy functional with respect to the standard deviation of a Gaussian trial wave–function. Our analytical results are in good agreement with the numerical calculations of Edward and Burnett[5)] and also of Dalfovo and Stringari[6)]. We are able to estimate the maximum number of bosons for which the BEC is stable. Moreover we find strong differences between the one–dimensional and the three–dimensional cases.

For the alkali vapors the range of the atom–atom interaction is believed to be short in comparison to the typical length scale of variations of atomic wave functions. The atom–atom interaction can be replaced by an effective zero–range interaction potential[5)]

$$U(\mathbf{r}-\mathbf{r}') = B\delta(\mathbf{r}-\mathbf{r}') \; . \tag{1}$$

Such an effective potential leads automatically to s–wave scattering only, where $B = 2\pi\hbar^2 a/m$ is the scattering amplitude and a is the s–wave scattering length. This scattering length is supposed to be positive for ^{87}Rb and ^{23}Na but negative for ^{7}Li. This means that for ^{87}Rb and ^{23}Na the interatomic interaction is repulsive while for ^{7}Li the atom–atom interaction is effectively attractive[3),10)].

By applying the theory of weakly interacting bosons[11], the Hamiltonian operator can be written

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r}) + V_m(\mathbf{r}) \; , \tag{2}$$

where $V_{ext}(\mathbf{r})$ is the external potential of the trap and V_m is the mean–field self–consistent potential, given by

$$V_m(\mathbf{r}) = \int d\mathbf{r}' \; |\psi(\mathbf{r}')|^2 U(\mathbf{r}-\mathbf{r}') \; , \tag{3}$$

where $\psi(\mathbf{r})$ is the wave–function of the BEC. Although the actual experimental traps are anisotropic[1,2,3], it is simplest to consider an isotropic harmonic trap (the effect of the anisotropy can be treated in perturbation theory). The bosons are alkali atoms in the trap

$$V_{ext}(\mathbf{r}) = \frac{m\omega^2}{2} r^2 \; , \tag{4}$$

with zero–range effective interaction, thus by using the equation (1) we get

$$V_m(\mathbf{r}) = B|\psi(\mathbf{r})|^2 \; . \tag{5}$$

The mean energy of the system is given by the Gross–Pitaevskii functional[12]

$$K[\psi] = \int d\mathbf{r} \; \psi^*(\mathbf{r})\hat{H}\psi(\mathbf{r}) \; , \tag{6}$$

and we can study the ground state stability by imposing the minimization of the energy functional

$$\delta K[\psi] = 0 \; . \tag{7}$$

The main point of this paper is the choice of the trial wave–function for the energy functional. We choose a Gaussian wave–function with a free

parameter σ, which is the standard deviation of the Gaussian, i.e. the mean radius of the condensate. In fact, for $\sigma = \sqrt{\frac{\hbar}{m\omega}}$ the test function is the ground–state function of the non–interacting system. Moreover, previous numerical results show that a Gaussian is a good approximation of the true ground–state function of the BEC[4),5),6)].

Let us start with the one–dimensional case. We choose the following trial wave–function

$$\psi(x) = C \exp{(\frac{-x^2}{2\sigma^2})} \; , \tag{8}$$

with the normalization condition

$$\int dx \; |\psi(x)|^2 = N \; , \tag{9}$$

where N is the number of bosons, from which we obtain

$$C^2 = \frac{N}{\pi^{1/2}\sigma} \; . \tag{10}$$

By inserting this trial wave–function in the energy functional, after some simple calculations, we find

$$K = \frac{1}{2}(\frac{\hbar^2}{2m})N\frac{1}{\sigma^2} + \frac{1}{2}(\frac{m\omega^2}{2})N\sigma^2 + \frac{BN^2}{(2\pi)^{1/2}}\frac{1}{\sigma} \; . \tag{11}$$

The minimum of the energy functional with respect to the standard deviation σ is obtained by imposing the following condition

$$0 = \frac{dK}{d\sigma} = -(\frac{\hbar^2}{2m})N\frac{1}{\sigma^3} + (\frac{m\omega^2}{2})N\sigma - \frac{BN^2}{(2\pi)^{1/2}}\frac{1}{\sigma^2} \; , \tag{12}$$

from which we get the formula

$$N = \frac{(2\pi)^{1/2}}{B}[(\frac{m\omega^2}{2})\sigma^3 - (\frac{\hbar^2}{2m})\frac{1}{\sigma}] \; . \tag{13}$$

It is easy to see that for any B there is only one solution of the equation $\frac{dK}{d\sigma} = 0$. The second derivative $\frac{d^2K}{d\sigma^2}$ is positive when $\frac{dK}{d\sigma} = 0$, namely the solution is stable, i.e. a minimum of the energy functional. In particular, for $N = 0$ we have $\sigma = \sqrt{\frac{\hbar}{m\omega}}$ and if $B > 0$ then $\sigma \to \infty$ for $N \to \infty$, while if $B < 0$ then $\sigma \to 0$ for $N \to \infty$.

Now we consider the three–dimensional case. We choose again a Gaussian trial wave–function

$$\psi(\mathbf{r}) = C \exp{(\frac{-r^2}{2\sigma^2})} \ , \tag{14}$$

with the normalization condition

$$\int d\mathbf{r} \ |\psi(\mathbf{r})|^2 = N \ , \tag{15}$$

from which we find

$$C^2 = \frac{N}{\pi^{3/2}\sigma^3} \ . \tag{16}$$

The resulting energy functional is slightly different from the one–dimensional one

$$K = \frac{3}{2}(\frac{\hbar^2}{2m})N\frac{1}{\sigma^2} + \frac{3}{2}(\frac{m\omega^2}{2})N\sigma^2 + \frac{BN^2}{(2\pi)^{3/2}}\frac{1}{\sigma^3} \ . \tag{17}$$

We find the minimum of the energy functional with respect to the mean radius σ by imposing the following condition

$$0 = \frac{dK}{d\sigma} = -3(\frac{\hbar^2}{2m})N\frac{1}{\sigma^3} + 3(\frac{m\omega^2}{2})N\sigma - 3\frac{BN^2}{(2\pi)^{1/2}}\frac{1}{\sigma^4} \ , \tag{18}$$

from which we obtain the formula

$$N = \frac{(2\pi)^{3/2}}{B}[(\frac{m\omega^2}{2})\sigma^5 - (\frac{\hbar^2}{2m})\sigma] \ . \tag{19}$$

By studying the function $\frac{dK}{d\sigma}$, we observe that for $B > 0$ there is only one solution (intersection with the σ–axis), which is a minimum of the energy

functional (stable solution), instead for $B < 0$ there are two solutions: one is a maximum (unstable solution) and the other a minimum (stable solution) of the energy functional.

It follows that for $B > 0$, when $N = 0$ we have $\sigma = \sqrt{\frac{\hbar}{m\omega}}$ and $\sigma \to \infty$ for $N \to \infty$. For $B < 0$ the mean radius σ decreases by increasing the number of bosons N to a minimum radius σ^{min}, with a maximum number N^{max} of bosons. For greater values of N the condensate becomes unstable and there is the collapse of the wave–function. It is not difficult to obtain the critical number of bosons. We put $\tilde{B} = -B$ and get

$$0 = \frac{dN}{d\sigma} = \frac{(2\pi)^{3/2}}{\tilde{B}}[(\frac{\hbar^2}{2m}) - 5(\frac{m\omega^2}{2})\sigma^4] \ , \tag{20}$$

from which we have the minimum radius

$$\sigma^{min} = \frac{1}{5^{1/4}}\sqrt{\frac{\hbar}{m\omega}} \ , \tag{21}$$

and the maximum number of bosons

$$N^{max} = \frac{4}{5^{5/4}}\frac{(2\pi)^{3/2}}{\tilde{B}}\frac{\hbar^2}{2m}\sqrt{\frac{\hbar}{m\omega}} \ . \tag{22}$$

Thus we obtain an analytical formula of the maximum number N^{max} of bosons for which the condensate, with attractive interaction, is stable. We have seen that in the one–dimensional case $N^{max} = \infty$ and $\sigma^{min} = 0$.

For ^7Li the scattering length is $a = -14.5$ Å, the axial frequency of the trap is $\omega_a/(2\pi) = 117$ Hz and the transverse frequency is $\omega_t/(2\pi) \simeq 163$ Hz (see Ref. 3 for further details). We estimate the critical number of particle by using $\omega/(2\pi) \simeq 120$ Hz for the frequency of our isotropic trap: we find $N^{max} \simeq 1400$. This result is in good agreement with the numerical

calculations of Dalfovo and Stringari[6]. It is important to observe the the number of particles in the BEC reported in the experimental work of Ref. 3 is an order of magnitude higher than our critical value. As suggested by Dalfovo and Stringari[6], the discrepancy between the experimental finding of Ref. 3 and the predictions of the Gross–Pitaevskii theory could be significantly reduced if one assumes the existence of a vortex in the atomic cloud[11].

In conclusion, we have studied in the mean–field approximation the ground–state stability of a gas of weakly interacting bosons in a harmonic trap with a zero–range interaction. To minimize the energy functional, we have chosen a Gaussian trial wave–function, where its standard deviation represents the mean radius of the condensate. In the case of repulsive interaction the mean radius of the condensate grows by increasing the number of bosons. In the case of attractive interaction the mean radius decreases by increasing the number of bosons but in different ways depending on the dimensionality. Our results suggest that one must be very careful when tries to simplify a many–body problem by reducing its dimension.

Our technique can be applied also to bosons with non–local interaction and to systems with different shapes of the trapping potential[13]. In the future will be interesting to investigate analytically the collective excitation of the condensate, at least in the semiclassical approximation[14,15].

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